A SOLUTION TO THE SPLITTING MIXED GROUP PROBLEM OF BAER

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1. **Introduction.** All groups considered in this paper are assumed to be additively written abelian groups. We follow for the most part the notation and terminology of [8]. We mention the following notations and definitions. If G is an abelian group, then tG denotes the maximal torsion subgroup of G and G_p denotes the primary component of tG for the prime p. The p-height of an element $x \in G$ is denoted by $h_p^G(x)$. We call a group G Hausdorff if it is Hausdorff in its n-adic topology. The group of rationals is denoted by Q and the subgroup of Q consisting of the integers by Z. The direct product of $\aleph_0$ copies of Z is denoted by P. After Nunke [11], a group G is called locally free if it is isomorphic to a pure subgroup of a direct product of copies of Z, that is, if G is $\aleph_1$-free and separable.

The central problem of this paper is one posed by Baer [1] in 1936 (also see Problem 30 in [8]) that asks for a characterization of those groups G for which $\operatorname{Ext}(G, T)=0$ for all torsion groups T. Such a group is called a Baer group or more simply a B-group. Baer [1] proved that any countable B-group is free. Since subgroups of B-groups are again B-groups, it follows that B-groups are $\aleph_1$-free. More recently Baer [3], Erdös [7] and Sasiada [8] independently proved that P is not a B-group. J. Rotman [12] showed that separable B-groups are slender and Nunke [11] removed the condition of separability. In this paper we settle the problem by showing that B-groups are necessarily free. More generally, for a nonempty subset N of the primes, we call a group G a B^N-group if $\operatorname{Ext}(G, T)=0$ for all torsion groups T such that $T_p=0$ for $p \notin N$. We are successful in finding necessary and sufficient conditions on the structure of a group G in order that G be a B^N-group for a prescribed subset N of the primes. The proofs of these results rest on the existence of a remarkable class of mixed groups. We show, for any cardinal number μ, that there is a Hausdorff mixed group M such that M/tM is divisible of rank μ and such that every torsion free subgroup of M is free. Our method of constructing such a group M is a somewhat complicated one. The most important tool used in showing that every torsion free subgroup of M is free is the "back-and forth" technique for direct sum decompositions introduced by Hill and Megibben in [10]. The final section of this paper deals with locally free B^N-groups (not all

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B^N-groups need be locally free). We show that if N is a nonempty proper subset of the primes, then there is a locally free B^N-group that is not free.

2. **An existence theorem concerning mixed groups.** We begin by establishing a special case of our main result on mixed groups.

LEMMA 2.1. *There is a Hausdorff mixed group X such that $X/tX \cong Q$ and such that any torsion free subgroup of X is cyclic.*

Proof. For each prime p, let $M^{(p)} = \prod_{n<\omega} \{b_{pn}\}$ where $\{b_{pn}\}$ denotes the cyclic group of order p^n and set $M = \prod_p M^{(p)}$ where p ranges over the primes. Let $u = \langle u_p \rangle \in M$ where u_p is an element of $M^{(p)}$ defined as follows: $u_p = \langle p^{e_n} b_{pn} \rangle$ where $e_n = [n/2]$ ([ ] denotes the greatest integer function). Note that $h_p^{M^{(p)}}(u_p) = 0$ since $e_1 = 0$. It is elementary to see that u_p is an element of infinite order in $M^{(p)}$ and hence that u is an element of infinite order in M. It is also easily seen, for each $y = \langle y_p \rangle \in tM$ and each prime p, that $h_p^M(nu + y) = h_p^{M^{(p)}}(nu_p + y_p) < \infty$ where n is a nonzero integer. To show that $h_p^{(M/tM)}(u + tM) = \infty$ for each prime p, it suffices to show that for each nonzero integer m and each prime p there is an element $x^{(m)} \in tM^{(p)}$ such that $u_p - x_p^{(m)} \in p^m M^{(p)}$. Clearly, there is a positive integer N such that if $n \geqq N$ then $e_n \geqq m$. Define $x_p^{(m)} = \langle \delta_n b_{pn} \rangle$ where $\delta_n = p^{e_n}$ for $n = 1, \ldots, N$ and $\delta_n = 0$ for $n > N$. Then $u_p - x_p^{(m)} \in p^m M^{(p)}$ and $x_p^{(m)} \in tM^{(p)}$. Since M/tM is torsion free there is a pure subgroup X of M containing tM and u such that X/tM is a pure rank one subgroup of M/tM. Since rank $(X/tM) = 1$ and since $h_p^{(X/tM)}(u + tM) = h_p^{(M/tM)}(u + tM) = \infty$ for each p, it follows that $X/tM = X/tX \cong Q$.

Suppose that A is a subgroup of X such that $A \cap tX = 0$. We may further suppose that $A \neq 0$. Since $A \cong \{A, tX\}/tX \subseteq X/tX \cong Q$, it follows that A is torsion free of rank one. Let $a \neq 0 \in A$. Since $X/tX \cong Q$ there are nonzero integers r and s such that $ra = su + y$ where $y = \langle y_p \rangle \in tX$. We have already observed that $h_p^X(ra) = h_p^X(su + y) = h_p^M(su + y) < \infty$ for each p. Hence $h_p^A(ra) < \infty$ for each prime p. There is a positive integer N such that $y_p = 0$ for all $p > N$. Let p be any prime larger than $N + |s|$. Then $(s, p) = 1$ and $h_p^A(ra) \leqq h_p^X(su + y) = h_p^{M^{(p)}}(su_p)$. Since s and p are relatively prime, $h_p^{M^{(p)}}(su_p) = h_p^{M^{(p)}}(u_p) = 0$. Thus, $h_p^A(ra) = 0$ for each prime p larger than $N + |s|$. Hence, the height sequence for ra in A is zero except for a finite number of primes and contains no "infinities". (For definition of height sequence, see [8].) It follows by a Theorem of Baer [2] that $A \cong Z$.

Let X be a group satisfying Lemma 1, μ a cardinal number and let Λ be an initial segment of the ordinal numbers having cardinality μ. For the remainder of this section, we define the group K_μ to be $\sum_{\lambda \in \Lambda} X_\lambda$ where $X_\lambda \cong X$ for each $\lambda \in \Lambda$. We now improve Lemma 2.1.

LEMMA 2.2. *Let A be a torsion free subgroup of K_μ. Then A is $\aleph_1$-free.*

Proof. By a theorem of Pontryagin [8] we need only show that every subgroup of A of finite rank is free. Since any subgroup of A of finite rank is isomorphic to a torsion free group contained in a finite number of the groups X_λ, it is enough to prove the lemma for K_n where n is a positive integer. For $n=1$, the lemma follows from Lemma 2.1. Therefore, suppose the result holds for all integers $\mu \leqq n$ and consider a torsion free subgroup A of $K_{n+1}=\sum_{i=1}^{n+1} X_i$. Let θ be the natural projection of K_{n+1} onto K_{n+1}/tK_{n+1} where θ restricted to X_i is the natural projection of X_i onto X_i/tX_i. Also let $B=\{a \in A \mid \theta(a) \in \sum_{i=1}^{n} \theta(X_i)\}$ and let π be the natural projection of K_{n+1} onto $\sum_{i=1}^{n} X_i$. Suppose that $b \in B \cap \operatorname{Ker} \pi = B \cap X_{n+1}$. Since $\theta(b) \in \sum_{i=1}^{n} \theta(X_i)$, it follows that $b \in tX_{n+1} \subseteq tK_{n+1}$. Therefore, $b=0$ and $B \cong \pi(B) \subseteq \sum_{i=1}^{n} X_i \cong K_n$. It follows from the induction hypothesis that B is free. Hence, $B=\sum_{j=1}^{k} \{b_j+y_j\}$ where $k \leqq n$, $b_j \in \sum_{i=1}^{n} X_i$ and $y_j \in X_{n+1}$. Since $\theta(b_j+y_j) \in \sum_{i=1}^{n} \theta(X_i)$, it follows that $y_j \in tX_{n+1}$ for $j=1, \ldots, k$. Let m be a positive integer such that $my_j=0$ for $j=1, \ldots, k$ and let ρ be the natural projection of K_{n+1} onto X_{n+1}. Set $\phi=m\rho$. It is easily verified that $A \cap \operatorname{Ker} \phi = B$ and that $\phi(A)$ is a torsion free subgroup of X_{n+1}. By Lemma 2.1, $\phi(A)$ is cyclic. Hence $A \cong B+\phi(A)$ and thus A is free.

We now establish the main result of this section. For notational convenience we use the notation $\sum_I X_\lambda$ to indicate $\sum_{\lambda \in I} X_\lambda$ where $I \subseteq \Lambda$.

THEOREM 2.3. *For any cardinal number μ, there is a Hausdorff mixed group M such that M/tM is divisible of rank μ and such that every torsion free subgroup of M is free.*

Proof. Let $M=K_\mu=\sum_\Lambda X_\lambda$. For the purpose of this proof, we assume that $0 \in \Lambda$ and that $X_0=0$. From the definition of K_μ, it is enough to show that every torsion free subgroup of K_μ is free. If μ is countable, the result follows from Lemma 2.2. Hence, we may assume that μ is uncountable. Let A be a torsion free subgroup of K_μ. Since $|K_\mu/tK_\mu|=\mu=|\Lambda|$, it follows that $|A| \leqq |\Lambda|$. Thus, we may label the elements of A with the ordinals in Λ starting with $a_0=0 \in A$ (we label $0 \in A$ repeatedly if necessary). Again let θ be the natural projection of K_μ onto K_μ/tK_μ. We wish to express A and a subset of Λ as unions of well-ordered monotone sequences $[A_\alpha]_{\alpha \in \Lambda}$ and $[I_\alpha]_{\alpha \in \Lambda}$, respectively such that $A_0=\{a_0\}=0$ and $I_0=[0]$ and such that

(i) $|I_{\alpha+1}-I_\alpha| \leqq \aleph_0$.

(ii) $I_\alpha=\bigcup_{\gamma<\alpha} I_\gamma$ and $A_\alpha=\bigcup_{\gamma<\alpha} A_\gamma$ if α is a limit ordinal.

(iii) $a_\alpha \in A_{\alpha+1}$.

(iv) $A_\alpha=A \cap (\sum_{I_\alpha} X_\lambda)=\{a \in A \mid \theta(a) \in \sum_{I_\alpha} \theta(X_\lambda)\}$.

Suppose that the I_α's and the A_α's satisfying (i)–(iv) have been chosen for all $\alpha<\beta$, $\beta \in \Lambda$. If β is a limit ordinal, then we need only set $I_\beta=\bigcup_{\alpha<\beta} I_\alpha$ and $A_\beta=\bigcup_{\alpha<\beta} A_\alpha$. We may assume that $\beta-1$ exists. We define sequences $[B_n]_{n<\omega}$ and $[L_n]_{n<\omega}$ inductively as follows: $B_1=\{A_{\beta-1}, a_{\beta-1}\}$ and let L_1 be the smallest subset of Λ such that $B_1 \subseteq \sum_{L_1} X_\lambda$. Clearly, $I_{\beta-1} \subseteq L_1$ and $|L_1-I_{\beta-1}| \leqq \aleph_0$. In general for $n>1$, we let L_n be the smallest subset of Λ such that $B_n=\{a \in A \mid \theta(a) \in \sum_{L_{n-1}} \theta(X_\lambda)\} \subseteq \sum_{L_n} X_\lambda$. Since $B_{n+1}=\{a \in A \mid \theta(a) \in \sum_{L_n} \theta(X_\lambda)\}$ and since $B_n \subseteq \sum_{L_n} X_\lambda$, it follows that $B_n \subseteq B_{n+1}$ and that $L_n \subseteq L_{n+1}$. Set $I_\beta=\bigcup_{n<\omega} L_n$ and set $A_\beta=\bigcup_{n<\omega} B_n$. Since

(ii) and (iii) clearly hold for $[I_\alpha]_{\alpha \leqq \beta}$ and $[A_\alpha]_{\alpha \leqq \beta}$, we need only verify (i) and (iv). If $a_{\beta-1} \in A_{\beta-1}$, it is easily seen that $A_\beta = A_{\beta-1}$ and $I_\beta = I_{\beta-1}$. Therefore, we may assume that $a_{\beta-1} \notin A_{\beta-1}$. To show that $|I_\beta - I_{\beta-1}| \leqq \aleph_0$, it suffices to show that $|L_n - I_{\beta-1}| \leqq \aleph_0$ for each n. Since we have already observed that $|L_1 - I_{\beta-1}| \leqq \aleph_0$, we suppose that $|L_n - I_{\beta-1}| \leqq \aleph_0$ and consider L_{n+1}. By definition of L_{n+1}, it is enough to show that $|B_{n+1}/A_{\beta-1}| \leqq \aleph_0$. We also may assume that $L_n \neq L_{n+1}$. Let π be the natural projection of K_μ onto $\sum_{L_n - I_{\beta-1}} X_\lambda$. Clearly, $A_{\beta-1} \subseteq \operatorname{Ker} \pi \cap B_{n+1}$. Suppose that $x \in B_{n+1}$ and that $mx \in \operatorname{Ker} \pi \cap B_{n+1}$ where m is a nonzero integer. Then $x = y + w$ where $y \in \sum_{I_{\beta-1}} X_\lambda$ and $w \in \sum_{L_{n+1} - I_{\beta-1}} X_\lambda$. Since $\pi(w) \in t(\sum_{L_n - I_{\beta-1}} X_\lambda)$ and since $\theta(x) \in \sum_{L_n} \theta(X_\lambda)$, we have that $w \in t(\sum_{L_{n+1} - I_{\beta-1}} X_\lambda)$ which implies that $\theta(x) = \theta(y) \in \sum_{I_{\beta-1}} \theta(X_\lambda)$. By (iv), $x \in A_{\beta-1}$. Hence, $\operatorname{Ker} \pi \cap B_{n+1} = A_{\beta-1}$ and $\pi(B_{n+1})$ is torsion free. Since $|L_n - I_{\beta-1}| \leqq \aleph_0$, it follows from the definition of the X_λ's that $|\pi(B_{n+1})| \leqq \aleph_0$. Thus $|B_{n+1}/A_{\beta-1}| \leqq \aleph_0$ and hence $|L_{n+1} - I_{\beta-1}| \leqq \aleph_0$. Thus (i) holds for $\alpha \leqq \beta$. Now if $x \in A \cap (\sum_{I_\beta} X_\lambda)$ then $x \in A \cap (\sum_{L_n} X_\lambda)$ for some n. Therefore, $\theta(x) \in \sum_{L_n} \theta(X_\lambda)$ which implies that $x \in B_{n+1} \subseteq A_\beta$. Since $A_\beta \subseteq A \cap (\sum_{I_\beta} X_\lambda)$, we have that $A_\beta = A \cap (\sum_{I_\beta} X_\lambda)$. We also have that $A_\beta = A \cap (\sum_{I_\beta} X_\lambda) \subseteq \{a \in A \mid \theta(a) \in \sum_{I_\beta} \theta(X_\lambda)\}$. If $\theta(a) \in \sum_{I_\beta} \theta(X_\lambda)$ where $a \in A$, then $\theta(a) \in \sum_{L_n} \theta(X_\lambda)$ for some n. By definition, $a \in B_{n+1} \subseteq A_\beta$. Hence,

$$A_\beta = A \cap \Big(\sum_{I_\beta} X_\lambda\Big) = \Big\{a \in A \mid \theta(a) \in \sum_{I_\beta} \theta(X_\lambda)\Big\}.$$

Thus $[A_\alpha]_{\alpha \leqq \beta}$ and $[I_\alpha]_{\alpha \leqq \beta}$ satisfy (i)–(iv).

We now establish

(v) A_α is a direct summand of $A_{\alpha+1}$ and $A_{\alpha+1} = A_\alpha + F_\alpha$ where F_α is free.

Let π_α be the natural projection of K_μ onto $\sum_{I_{\alpha+1} - I_\alpha} X_\lambda$ (we may assume that $I_{\alpha+1} \neq I_\alpha$). Suppose that $x \in A_{\alpha+1}$ such that $\pi(x) \in t(\sum_{I_{\alpha+1} - I_\alpha} X_\lambda)$. Then $x = y + w$ where $y \in \sum_{I_\alpha} X_\lambda$ and where $w \in t(\sum_{I_{\alpha+1} - I_\alpha} X_\lambda)$. This implies that $\theta(x) = \theta(y) \in \sum_{I_\alpha} \theta(X_\lambda)$ which implies by (iv) that $x \in A_\alpha$. Hence, $A_{\alpha+1} \cap \operatorname{Ker} \pi_\alpha = A_{\alpha+1} \cap (\sum_{I_\alpha} X_\lambda) = A \cap (\sum_{I_\alpha} X_\lambda) = A_\alpha$ and $\pi_\alpha(A_{\alpha+1})$ is torsion free. Therefore, $A_{\alpha+1}/A_\alpha$ is isomorphic to a torsion free subgroup of $\sum_{I_{\alpha+1} - I_\alpha} X_\lambda$. Since by (i) $|I_{\alpha+1} - I_\alpha| \leqq \aleph_0$ and since $|K_{\aleph_0}/tK_{\aleph_0}| \leqq \aleph_0$, it follows that $|A_{\alpha+1}/A_\alpha| \leqq \aleph_0$. By Lemma 2, $A_{\alpha+1}/A_\alpha$ is free. Thus (v) is established. Since A_0 is free, then (ii) and (v) imply that A is free.

3. **The structure of B^N-groups.** Let G be a torsion free group and let T be a torsion group. Then an extension $T \rightarrowtail H \twoheadrightarrow G$ is called a quasi-splitting extension of T by G if H is quasi-isomorphic to $T + G$. By a Theorem of C. Walker [14], the extension above is quasi-splitting if and only if it represents an element of finite order in $\operatorname{Ext}(G, T)$. The following theorem characterizes those torsion free groups G for which every extension $T \rightarrowtail H \twoheadrightarrow G$ is quasi-splitting for all torsion groups T.

THEOREM 3.1. *Let G be a torsion free group. Then* $\operatorname{Ext}(G, T)$ *is torsion for all torsion groups T if and only if G is free.*

Proof. The sufficiency is clear. Therefore, suppose that G is a torsion free group such that $\operatorname{Ext}(G, T)$ is torsion for all torsion groups T. Let $\mu = \operatorname{rank}(G)$ and let M

be a group satisfying Theorem 2.3 such that rank $(M/tM)=\mu$. Since M/tM is torsion free and divisible there is a monomorphism $f: G \rightarrowtail M/tM$. Let θ be the natural map of M onto M/tM. The exactness of the sequence

$$\text{Hom}(G, M) \xrightarrow{\theta_*} \text{Hom}(G, M/tM) \xrightarrow{\delta_G} \text{Ext}(G, tM)$$

implies that there is a nonzero integer n such that $nf \in \text{Im } \theta_*$, that is, there is a homomorphism $\phi \in \text{Hom}(G, M)$ such that $nf = \theta\phi$. Since nf is also a monomorphism, ϕ must be a monomorphism and $\phi(G) \cap \text{Ker } \theta = \phi(G) \cap tM = 0$. Thus G is isomorphic to a torsion free subgroup of M which implies by Theorem 2.3 that G is free.

The following corollary settles the question of Baer that was mentioned in the introduction.

COROLLARY 3.2. *A group G is a Baer group if and only if it is free.*

Proof. Again the sufficiency is clear. Since Baer groups are necessarily torsion free (see [1]) we may apply Theorem 3.1 to prove the necessity.

Before continuing, one should observe that our results are valid for modules over a principal ideal domain(2). Let N be a nonempty subset of the primes and let I_N be the subring of the rationals Q defined by the rule: $m/n \in Q$, where $m, n \in Z$ and $n \neq 0$, is an element of I_N if and only if n and p are relatively prime for each prime $p \in N$. We also use the symbol I_N to denote the additive group of I_N. However, no confusion should arise. Observe that a torsion group T, where $T_p = 0$ for $p \notin N$, can be considered a module in a natural fashion over the ring I_N. For a group G, one should also observe that $\text{Hom}(I_N \otimes G, T) = \text{Hom}_{I_N}(I_N \otimes G, T)$(3). We now establish the following lemma.

LEMMA 3.3. *Let N be a nonempty subset of the primes and let T be a torsion group such that $T_p = 0$ for $p \notin N$. If G is a group such that* $\text{Tor}(G, I_N/Z) = 0$, *then* $\text{Ext}(G, T)$, $\text{Ext}(I_N \otimes G, T)$ *and* $\text{Ext}_{I_N}(I_N \otimes G, T)$ *are isomorphic as abelian groups.*

Proof. We may assume that N is a proper subset of the primes since otherwise $I_N = Z$. Let $\tilde{N}$ be the set of primes not in N. From the definition of I_N we obtain the exact sequence $Z \rightarrowtail I_N \twoheadrightarrow \sum_{p \in \tilde{N}} C(p^\infty)$ which yields the exact sequence $Z \otimes G \rightarrowtail I_N \otimes G \twoheadrightarrow \sum_{p \in \tilde{N}} C(p^\infty) \otimes G$. Hence, we obtain the exact cohomology sequence $\text{Ext}(\sum_{p \in \tilde{N}} C(p^\infty) \otimes G, T) \rightarrowtail \text{Ext}(I_N \otimes G, T) \twoheadrightarrow \text{Ext}(Z \otimes G, T)$. Since $\sum_{p \in \tilde{N}} C(p^\infty) \otimes G$ and T are torsion groups with no nonzero primary components in common, we have that $\text{Ext}(\sum_{p \in \tilde{N}} C(p^\infty) \otimes G, T) = 0$ and thus

$$\text{Ext}(I_N \otimes G, T) \cong \text{Ext}(Z \otimes G, T) \cong \text{Ext}(G, T).$$

(2) All rings considered in this paper are assumed to be commutative with identity and all modules are assumed to be unital.

(3) We drop the subscript "R" on the functors $\text{Hom}_R(A, B)$ and $\text{Ext}_R(A, B)$ only when $R = Z$.

Let E be the I_N-injective envelope of T. Clearly E is both injective as an I_N-module and as an abelian group. Therefore, the exactness of $T \rightarrowtail E \twoheadrightarrow E/T$ induces exactness of the rows of the commutative diagram:

$$\begin{array}{ccccccc} \operatorname{Hom}(I_N \otimes G, T) & \to & \operatorname{Hom}(I_N \otimes G, E) & \to & \operatorname{Hom}(I_N \otimes G, E/T) & \twoheadrightarrow & \operatorname{Ext}(I_N \otimes G, T) \\ \| & & \| & & \| & & \\ \operatorname{Hom}_{I_N}(I_N \otimes G, T) & \to & \operatorname{Hom}_{I_N}(I_N \otimes G, E) & \to & \operatorname{Hom}_{I_N}(I_N \otimes G, E/T) & \twoheadrightarrow & \operatorname{Ext}_{I_N}(I_N \otimes G, T) \end{array}$$

Thus the cokernels are isomorphic.

THEOREM 3.4. *A group G is a B^N-group if and only if $I_N \otimes G$ is free as an I_N-module.*

Proof. Suppose that G is a B^N-group. Then clearly $(tG)_p = 0$ for each $p \in N$. Therefore, $I_N \otimes (tG) = 0$ and hence $I_N \otimes G \cong I_N \otimes (G/tG)$. The exactness of $0 = \operatorname{Hom}(tG, T) \to \operatorname{Ext}(G/tG, T) \to \operatorname{Ext}(G, T) = 0$, when T is a torsion group such that $T_p = 0$ for $p \notin N$, implies that G/tG is a B^N-group. By Lemma 3.3, $I_N \otimes (G/tG)$ is a Baer module for the ring I_N. Hence, $I_N \otimes (G/tG)$ is free as an I_N-module and thus $I_N \otimes G$ is also free as an I_N-module. With the aid of Lemma 3.3 the sufficiency is easily obtained.

A corollary to Theorem 3.4 is the following:

COROLLARY 3.5. *G is a B^N-group if and only if $(tG)_p = 0$ for each $p \in N$* and *G/tG is isomorphic to a subgroup of $\sum_\mu I_N$ where $\mu = \text{rank}\,(G/tG)$.*

Proof. By Theorem 3.4, G is a B^N-group if and only if $I_N \otimes G \cong \sum_\mu I_N$ for some cardinal number μ. Therefore, $I_N \otimes (tG) = 0$, which holds if and only if $(tG)_p = 0$ for $p \in N$. Hence $I_N \otimes G \cong I_N \otimes (G/tG)$. From the exactness of $Z \rightarrowtail I_N \twoheadrightarrow I_N/Z$, we obtain $G/tG \cong Z \otimes (G/tG) \rightarrowtail I_N \otimes (G/tG) \twoheadrightarrow (I_N/Z) \otimes (G/tG)$. Since $(I_N/Z) \otimes (G/tG)$ is torsion, we have that $\mu = \text{rank}\,(I_N \otimes (G/tG)) = \text{rank}\,(G/tG)$.

Our next corollary is an immediate consequence of Corollary 3.5 and a Theorem of Nunke [11] on slender groups.

COROLLARY 3.6. *A torsion free B^N-group is slender.*

4. **On locally free B^N-groups.** Let N be a nonempty proper subset of the primes. Although Theorem 3.4 implies there are nonfree B^N-groups (for example the group I_N), one might suspect that locally free B^N-groups are free. However, we shall presently show, for each nonempty proper subset N of the primes, that there is a pure subgroup of P and hence a locally free group which is a nonfree B^N-group. We begin by first generalizing a method of Chase [5] for constructing pure subgroups of P with certain prescribed properties. Let S denote the group of finite sequences in P and, for the purposes of our next lemma and theorem, let E denote the cotorsion completion of S. (For information concerning cotorsion groups, see [9].)

LEMMA 4.1. *Let C be a countable pure subgroup of P that contains S and let U be a pure subgroup of E. Then there is a pure subgroup A of P such that A contains C and $A/C \cong U$.*

Proof. Specker [13] has shown that P contains a pure free subgroup of rank $\aleph_1$. It follows by a Theorem of Chase [6] on pure independence, that any maximal pure independent subset of P has cardinality at least $\aleph_1$. Hence, there must be a subgroup F of P such that $C \subseteq F$ and such that F/C is a pure free subgroup of P/C of rank $\aleph_0$. Since P/S is cotorsion (see [11]) and since C is a pure subgroup of P containing S, then P/C is also a torsion free cotorsion group. Therefore, $P/C = K/C + \bar{C}/C$ where K/C is Hausdorff, $F \subseteq K$ and $\bar{C}/C$ is divisible. Let H be the subgroup of P such that H/C is the n-adic closure of the pure free subgroup F/C in K/C. Since $(K/C)/(H/C)$ must be reduced and torsion free, it follows that H/C is cotorsion and that H/C is a direct summand of K/C. Since F/C is pure and dense in H/C, we have that $H/C \cong E$. Therefore, the group U may be identified with a pure subgroup of P/C. Thus, there is a pure subgroup A of P such that A contains C and $A/C \cong U$.

THEOREM 4.2. *Let $[U_\alpha]_{\alpha<\Omega}$ be a family of countable pure subgroups of E. Then there is a pure subgroup A of P such that $A = \bigcup_{\alpha<\Omega} A_\alpha$ where the subgroups A_α satisfy:*

(i) *If $\alpha < \beta$, $A_\alpha \subseteq A_\beta$.*

(ii) *A_α is free of rank $\aleph_0$.*

(iii) *$A_{\alpha+1}/A_\alpha \cong U_\alpha$.*

(iv) *If α is a limit ordinal, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.*

Furthermore, if there is a countable ordinal λ such that $U_\alpha \neq 0$ and Hom $(U_\alpha, Z) = 0$ *for all $\alpha > \lambda$, then A is not free.*

Proof. The construction of a pure subgroup A of P satisfying (i)–(iv) is an immediate consequence of Lemma 4.1. Hence, suppose that there is a countable ordinal λ such that $U_\alpha \neq 0$ and Hom $(U_\alpha, Z) = 0$ for all $\alpha > \lambda$. Suppose that $f \neq 0 \in$ Hom (A, Z) such that $f(A_{\lambda+1}) = 0$. Let α be the smallest ordinal such that $f(A_\alpha) \neq 0$. Clearly, α cannot be a limit ordinal. Hence, $\alpha = \beta + 1$ and $\beta \geqq \lambda + 1$ since $f(A_{\lambda+1}) = 0$. Let h be the restriction of f to $A_{\beta+1}$. Since $h \neq 0$ and since $h(A_\beta) = 0$, it follows that there is a nonzero homomorphism in Hom $(A_{\beta+1}/A_\beta, Z) \cong$ Hom (U_β, Z). But this is a contradiction since $\beta \geqq \lambda + 1 > \lambda$. Thus, if $f \in$ Hom (A, Z) and $f(A_{\lambda+1}) = 0$ then $f = 0$, that is, Hom $(A/A_{\lambda+1}, Z) = 0$. Since $U_\alpha \neq 0$ for all $\alpha > \lambda$, we have that $|A| = \aleph_1$. Clearly, no free group F of cardinality $\aleph_1$ has the property that Hom $(F/F_0, Z) = 0$ where F_0 is a countable subgroup of F. Thus A is not free.

THEOREM 4.3. *Let N be a nonempty proper subset of the primes. Then there is a locally free B^N-group that is not free.*

Proof. Observe that the cotorsion completion of the group I_N is a direct summand of E. Let A be the group constructed in Theorem 4.2 with $U_\alpha = I_N$ for each $\alpha < \Omega$. Since Hom $(I_N, Z) = 0$, we have that A is a nonfree, locally free group. Note that $I_N \otimes A = \bigcup_{\alpha<\Omega} (I_N \otimes A_\alpha)$. The exact sequence $A_\alpha \rightarrowtail A_{\alpha+1} \twoheadrightarrow I_N$ yields the exact sequence $I_N \otimes A_\alpha \rightarrowtail I_N \otimes A_{\alpha+1} \twoheadrightarrow I_N \otimes I_N$. By a Theorem of Baer [2], it is easily seen that $I_N \otimes I_N \cong I_N$. Since, for each α, $I_N \otimes A_\alpha$ is free as an I_N-module and since $(I_N \otimes A_{\alpha+1})/(I_N \otimes A_\alpha) \cong I_N$, it follows that $I_N \otimes A$ is free as an I_N-module. By Theorem 3.4, A is a B^N-group.

COROLLARY 4.4. *If μ is an uncountable cardinal and if N is a nonempty proper subset of the primes, then the completely decomposable group $\sum_\mu I_N$ contains a locally free group that is not completely decomposable.*

Proof. We may assume that $\mu = \aleph_1$. Let A be a B^N-group satisfying Theorem 4.3. Then the exact sequence $Z \rightarrowtail I_N \twoheadrightarrow I_N/Z$ yields that $A \cong Z \otimes A \rightarrowtail I_N \otimes A \cong \sum_{\aleph_1} I_N$.

If the set N consists of a single prime p, let $B^{(p)}$ denote B^N. Note that the statement that G is a $B^{(p)}$-group for each prime p does not imply the statement that G is a Baer group. Indeed, in view of Corollary 3.2, Chase [5] constructed a group that is a $B^{(p)}$-group for each p but that is not a Baer group. We also remark that if G is a $B^{(p)}$-group for each p, then Corollary 3.5 implies that G is torsion free. Our concluding theorem shows that if a group G is a $B^{(p)}$-group for each prime p then, for any torsion group T, a quasi-splitting extension of T by G is necessarily a splitting extension.

THEOREM 4.5. *If G is a $B^{(p)}$-group for each prime p, then* Ext (G, T) *is torsion free for any torsion group T.*

Proof. Let p be a prime. Then $T = H + T_p$ where $H_p = 0$. Therefore, Ext (G, T) $\cong$ Ext (G, H) + Ext (G, T_p) = Ext (G, H) since Ext $(G, T_p) = 0$. By a theorem in [4], we have the isomorphism

$$\text{Hom}\,(C(p), \text{Ext}\,(G, H)) + \text{Ext}\,(C(p), \text{Hom}\,(G, H))$$
$$\cong \text{Ext}\,(C(p) \otimes G, H) + \text{Hom}\,(\text{Tor}\,(C(p), G), H)$$

where $C(p)$ is the cyclic group of order p. Since $C(p) \otimes G$ is p-primary and since $H_p = 0$, then Ext $(C(p) \otimes G, H) = 0$. Also Hom (Tor $(C(p), G), H) = 0$ since G is torsion free. Thus Hom $(C(p)$, Ext $(G, T)) \cong$ Hom $(C(p)$, Ext $(G, H)) = 0$. Since p was an arbitrary prime, it follows that Ext (G, T) is torsion free.

REFERENCES

1. R. Baer, *The subgroup of the elements of finite order of an Abelian group*, Ann. of Math. **37** (1936), 766–781.

2. ———, *Abelian groups without elements of finite order*, Duke Math. J. **3** (1937), 68–112.

3. ———, *Die Torsionsuntergruppe einer Abelschen Gruppe*, Math. Ann. **135** (1958), 219–234.

4. H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

5. S. Chase, *Function topologies on Abelian groups*, Illinois J. Math. **7** (1963), 593–608.

6. ———, "On group extensions and a problem of J. H. C. Whitehead," in *Topics in Abelian groups*, Scott, Foresman and Co., Chicago, Ill., 1963, pp. 173–193.

7. J. Erdös, *On the splitting problem of mixed Abelian groups*, Publ. Math. Debrecen **5** (1958), 364–367.

8. L. Fuchs, *Abelian groups*, Hungarian Academy of Sciences, Budapest, 1958.

9. D. K. Harrison, *Infinite Abelian groups and homological methods*, Ann. of Math. **69** (1959), 366–391.

10. P. Hill and C. Megibben, *Extending automorphisms and lifting decompositions in Abelian groups*, Math. Ann. **175** (1968), 159–168.

11. R. J. Nunke, *Slender groups*, Acta. Sci. Math. Szeged **23** (1962), 67–73.

12. J. Rotman, *On a problem of Baer and a problem of Whitehead in Abelian groups*, Acta. Math. Acad. Sci. Hungar. **12** (1961), 245–254.

13. E. Specker, *Additive gruppen von Folgen ganzer Zahlen*, Portugal. Math. **9** (1950), 131–140.

14. C. Walker, *Properties of Ext and quasi-splitting of Abelian groups*, Acta. Math. Acad. Sci. Hungar. **15** (1964), 157–160.

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